Mid-Semester Exam ISI Bangalore 24th Feb 2012

Representation Theory of finite groups 100 Points

Notes.
(a) Begin each answer on a separate sheet.
(b) Justify all your steps. Assume only those theorems that have been proved in class. All other steps should be justified.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

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1. [5 points] Let V, W be vector spaces over a field k having basis $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ respectively. Express

$$(2e_1 + e_2 + e_3) \otimes (f_1 + f_2 + 2f_3) \quad + \quad (e_1 - e_2 + e_3) \otimes (f_1 - f_2 + f_3)$$

as a linear combination of the basis $\{e_i \otimes f_j\}_{i,j}$ of $V \otimes_k W$.

2. [15 points] Let V, W be vector spaces over a field k having basis $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ respectively. Let $u = \Sigma c_{ij}(e_i \otimes f_j)$ be an element of $V \otimes_k W$ with $c_{ij} \in k$.
   (i) Prove that if u is a pure tensor, i.e., there exist $v \in V$ and $w \in W$ such that $u = v \otimes w$, then $c_{ij}c_{kl} = c_{il}c_{kj}$.
   (ii) Prove the converse: To simplify the presentation, assume $c_{11} = 1$ and prove that if the above relations hold among the coefficients, then u is a pure tensor.

3. [14 points] Let $f\colon A \to B$ be a homomorphism of commutative rings and let M, N be B-modules. Thus we may consider M, N to be A-modules in the obvious natural way.
   (i) Show that there is a natural map of A-modules $\pi\colon M \otimes_A N \to M \otimes_B N$ which is surjective. (Note: Do you have a formula for π ?)
   (ii) In the situation of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ verify whether $i \otimes 1 = 1 \otimes i$ in any of these two modules where $i = \sqrt{-1}$.

4. [10 points] Let $f\colon A \to B$ be a homomorphism of commutative rings. Let M be an A-module and N a B-module so that N may also be considered to be an A-module in a natural way. Find a natural isomorphism $(M \otimes_A B) \otimes_B N \xrightarrow{\sim} M \otimes_A N$.
(Hint: Use some of the isomorphisms proved in class.)

5. [12 points] Let V be a vector space over a field k with basis e_1, e_2, e_3, e_4. Let $u = \Sigma a_i e_i$ and $v = \Sigma b_j e_j$ be elements of V where $a_i, b_j \in k$.

(i) Write a basis of $\mathrm{Sym}^2(V)$.
(ii) Express $u \cdot v$ in your basis above.
(iii) Write a basis of $\mathrm{Alt}^2(V)$.
(iv) Express $u \wedge v$ in your basis above.

6. [10 points] Let V be a vector space over a field k where $\mathrm{char}(k) \neq 2$. Let $u, v, w \in V$. Simplify the following products in the exterior algebra of V.

(i) $((u \wedge v) + w) \wedge ((u \wedge v) + w)$, (ii) $((u \wedge v) + w) \wedge ((u \wedge w) + v)$.

7. [12 points] Let k be a field and let $R = \mathrm{M}_2(k)$ the ring of 2×2 matrices over k. Let $I \subset R$ be a proper nonzero left ideal, i.e., $(0) \neq I \neq R$. Prove that there exists a unit $u \in R$ (i.e., u is an invertible matrix) such that Iu equals the left ideal consisting of matrices whose second column is zero. (Hint. Right multiplication by u amounts to performing column operations.)

8. [10 points] Let k be a field and $R = k[X]$, the polynomial ring in one variable. Let M be an R-module. Prove that M is simple (irreducible) if and only if $M \xrightarrow{\sim} k(\alpha)$ where α is an element in some finite field extension $k \to K$.

9. [12 points] Let k be an algebraically closed field of characteristic 0. Let G be a finite group. Use Artin-Wedderburn Theorem and semi-simplicity of group rings over k to prove the following.

(i) G is abelian if and only if there is an isomorphism of algebras $k[G] \xrightarrow{\sim} k \times k \times \cdots \times k$.
(ii) If $G = S_3$, then $k[G] \xrightarrow{\sim} k \times k \times \mathrm{M}_2(k)$. (Hint: Write 6 as a sum of squares.)